Limits at Infinity & Asymptotes:

Let $f(x) = 2 + \frac{1}{x}$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(2+\frac{1}{x}\right) = 2 + \lim_{x\to\infty}\frac{1}{x} = 2+0 = 2$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(2+\frac{1}{x}\right) = 2 + \lim_{x\to-\infty}\frac{1}{x} = 2-0 = 2$$

finite limit at infinity.

Horizontal Asymptote.

Let $f(x) = x^3$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^3 = \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3 = -\infty$$

infinite limit at infinity.

Def^n:- If f has limit at infinity if $\begin{cases} \lim\limits_{x\to\infty} f(x) = L & \text{or} \\ \lim\limits_{x\to-\infty} f(x) = L \end{cases}$

We call $y = L$ line, the Horizontal asymptote of f.

Def^n:- If $\left\{\begin{array}{l} \lim\limits_{x\to\infty} f(x) = +\infty \text{ or } -\infty \\ \lim\limits_{x\to-\infty} f(x) = +\infty \text{ or } -\infty \end{array}\right\}$ then f has an infinite limit at infinity.

Let $f(x) = 2 + \frac{1}{x}$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(2+\frac{1}{x}\right) = 2 + \lim_{x\to 0^+}\frac{1}{x} = 2+\infty = \infty$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(2+\frac{1}{x}\right) = 2 + \lim_{x\to 0^-}\frac{1}{x} = 2-\infty = -\infty$$

Def^n: If for a function f any of the following holds,

(1) $\lim_{x \to a} f(x) = \pm\infty$ or

(2) $\lim_{x \to a^+} f(x) = \pm\infty$ or

(3) $\lim_{x \to a^-} f(x) = \pm\infty$

then we call $x = a$ line to be the vertical asymptote of f.

- Find the asymptotes for $y = x^k$, where k is an integer.

Case 1: $k > 0$, $f(x) = x^k$ = monomial / polynomials

Then $$\lim_{x \to \pm\infty} x^k = \left.\begin{cases} +\infty, & \text{if } k \text{ is even} \\ -\infty, & \text{if } k \text{ is odd} \end{cases}\right\}$$ but not a finite value.

Therefore, $f(x) = x^k$, $k > 0$ has no horizontal asymptote.

Also, $\lim_{x \to a^+ / a^- / a} x^k = a^k$ (a finite value)

So, $f(x) = x^k$, $k > 0$ has no vertical asymptote.

Case 2: $k = 0$, $f(x) =$ constant ie, a horizontal line.

Clearly, it doesn't have vertical asymptote.

Also, it is itself the horizontal line, that was supposed to be the Horizontal Asymptote; so no horizontal asymptote by definition.

Case 3: $k < 0 \Rightarrow k = -r,\ r > 0$

So, $f(x) = x^k = x^{-r} = \dfrac{1}{x^r},\ r > 0.$

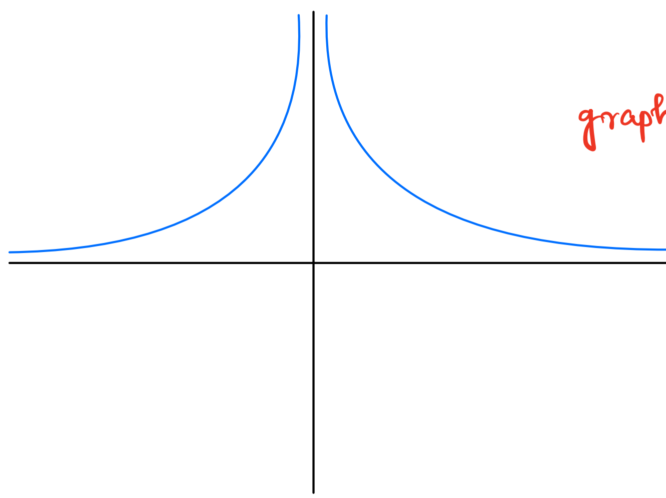
Note that $\displaystyle\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x^r} = 0$

So, $y = 0$ is our horizontal asymptote.

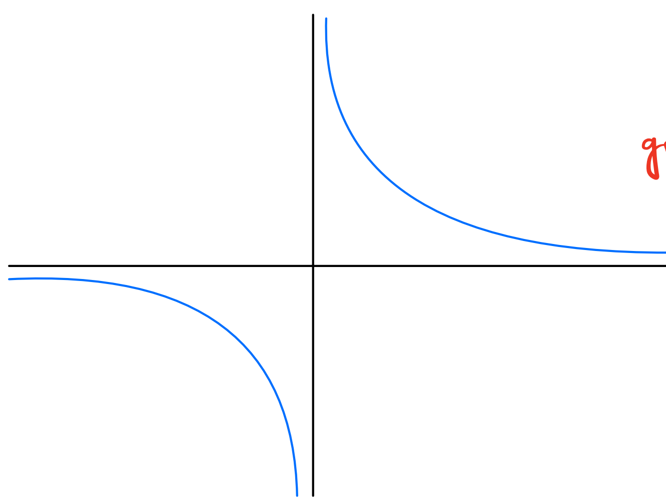
Also, $\displaystyle\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x^r} = +\infty$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x^r} = \begin{cases} +\infty, & \text{if } r \text{ is even} \\ -\infty, & \text{if } r \text{ is odd.} \end{cases}$$

So, $x = 0$ is the vertical asymptote of $f(x) = x^{-k}, k > 0$.



graph of $f(x) = x^{-k}; k > 0$ & k even.



graph of $f(x) = x^{-k}; k > 0$ & k odd

- Find the asymptotes for $f(x) = \dfrac{x^2+x-6}{2x^2+6}$.

Since, $2x^2+6 \geq 6$ for any real x, therefore the denominator is never 0

Hence, there is not possibility of getting a vertical asymptote.

Now, $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^2+x-6}{2x^2+6}$

$$= \lim_{x\to\infty} \frac{1+\frac{x}{x^2}-\frac{6}{x^2}}{2+\frac{6}{x^2}}$$

$$= \frac{1+0-0}{2+0} = \frac{1}{2}$$

And therefore, $y=\frac{1}{2}$ is the horizontal asymptote.

- Find the asymptotes for $f(x) = \dfrac{\sqrt{6x^2+7}}{x+7}$

Note: denominator is 0 when $x=-7$.

So if we choose $\lim\limits_{x\to -7} f(x) = \lim\limits_{x\to -7} \dfrac{\sqrt{6x^2+7}}{x+7} = \infty$

Hence, $x=-7$ is the Vertical Asymptote.

Also, $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\sqrt{6x^2+7}}{x+7}$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2\left(6+\frac{7}{x^2}\right)}}{x+7}$$

$$= \lim_{x\to\infty} \frac{x\sqrt{6+\frac{7}{x^2}}}{x\left(1+\frac{7}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{6+\frac{7}{x^2}}}{1+\frac{7}{x}}$$

$$= \frac{\sqrt{6+0}}{1+0} = \sqrt{6}.$$

Hence, $y=\sqrt{6}$ is a Horizontal Asymptote.